Finite Model Theory: A Personal Perspective

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"We can see further, by standing on the shoulders of giants"
– Bernard of Chartres (12C.)

Dr. Ronald Fagin

From Ron's CV:

- Ph.D. in Mathematics, University of California at Berkeley, 1973
- Thesis: "Contributions to the Model Theory of Finite Structures"
- Advisor: Prof. Robert L. Vaught
- National Science Foundation Graduate Fellowship 1967-72
- Research Assistantship 1972-73
- Passed Ph.D. Qualifying Exams .With Distinction. (top 5%).

“Contributions to the Model Theory of Finite Structures”

Foundations of Finite-Model Theory: *Three Seminal Results*

- Generalized first-order spectra and polynomial-time recognizable sets. In *Complexity of Computation*, ed. R. Karp, SIAM-AMS Proceedings 7, 1974, pp. 43–73.
- Monadic generalized spectra. *Zeitschr. f. math. Logik und Grundlagen d. Math.* 21, 1975, pp. 89–96
- Probabilities on finite models. *J. Symbolic Logic* 41:1, 1976, pp. 50–58.

Finite-Model Theory

Model Theory: $models(\varphi) = \{M \ : \ M \models \varphi\}$

- φ – 1st-order sentence
- M – structure

Example: "all graph nodes have at least two distinct neighbors"

$$(\forall x)(\exists y)(\exists z)(\neg(y = z) \wedge E(x, y) \wedge E(x, z))$$

Finite-Model Theory: focus on *finite* structures!

Descriptive-Complexity Theory

A complexity-theoretic perspective on finite-model theory:

- Fix φ and consider $models(\varphi)$ as a decision problem:

– Given M, does it satisfy φ, i.e., does $M \models \varphi$ hold?

Q: What is the complexity? (*Data Complexity*)

A: In LOGSPACE (easy!)

Existential Second-Order Logic (ESO)

Syntax: $(\exists R_1)\ldots(\exists R_k)\varphi$

- φ – first order

Semantics: Σ_1^1

- $\{models(\psi) : \psi \in ESO\}$

Data Complexity: NP – guess quantified relations $R_1 \ldots R_k$ and check that φ holds

Fagin's Theorem

Just observed: $\Sigma_1^1 \subseteq NP$

Fagin, 1974: $\Sigma_1^1 = NP$

- In words: Σ_1^1 *captures* NP

Amazing Result!

- No Turing machine
- No time
- No polynomial
- Pure logic!

Why Second-Order Logic?

Vardi, 1981: Why second-order logic?

- To simulate *nondeterminism*.
- To simulate *a linear order*, so we can count TM steps.

What if we:

- focus on deterministic machines, i.e., P instead of NP.
- assume that the structure comes with a built-in linear order.

The Immerman-Vardi Theorem

Chandra+Harel, 1980: **Fixpoint Logic** – augmenting first-order logic with bounded iteration:
$R \leftarrow \varphi(R, Q_1, \ldots, Q_m)$

- where R occurs *positively* in φ.

Theorem [Immerman, V., 1982]: Fixpoint Logic captures P on ordered structures.

- A logical characterization of P.

Major Open Question: Is there a logic that captures P *without* assuming built-in order. [Chandra+Harel, 1980]

Pure Descriptive-Complexity Theory

Computational-Complexity Theory: What *computational* resources are required to solve computational problems?

- **Example**: What is the computational complexity class of Digraph Reachability? NLOGSPACE!

Descriptive-Complexity Theory: What *logical* resources are required to solve computational problems?

- **Example**: What logic can express digraph reachability?

The Logic of Digraph Reachability

Observation: REACH is in P.

- **Consequence**: REACH is in both Σ_1^1 and Π_1^1.

Observation: REACH is in Monadic Π_1^1.

- **Question**: Is REACH in Monadic Σ_1^1

Fagin, 1975: REACH is *not* in Monadic Σ_1^1.

- **Corollary**: REACH is *not* in FO.
- Rediscovred by Aho+Ullman, 1978.

Built-In Relations

Major Issue in Descriptive-Complexity Theory: power of *built-in relations*

- **Example**: See Immerman-Vardi Theorem!

Question: What happens to $REACH \not\in Monadic\Sigma^1_1$ when we add built-in relations?

Theorem [Fagin-Stockmeyer-V., 1995]: $REACH \not\in Monadic\Sigma^1_1$ even when we add built-in relations of *moderate* degree, e.g., successor relation.

- Schwentick, 1996: even with linear order.

Fagin'74 vs Fagin'75

- Fagin'75: standard result in mathematical logic – Property X cannot be expressed in logic Y
  - **But**: restriction to finite structures makes the result more difficult.
- Fagin'74: Finiteness enables us to view a logical problem as a decision problem – yields connection to computational-complexity theory

Perspective: Finiteness opens the door to completely *new* questions in model theory!

Logical Validity

Validity: truth in in *all* structures – logical truth!

- The most fundamental notion in logic!

Finite Validity: truth in in *all finite* structures

But:

- Validity is semidecidable – Gödel
- Finite validity is not semidecidable – Trakhtenbrot

Almost-Sure Validity

Fagin'76: *Almost-Sure Validity* – truth over *almost all finite* structures

- Leverage finiteness to define limit probability

0-1 Law for First-Order Logic: For every first-order sentence φ, either φ or $\neg\varphi$ is almost-surely valid.

- A proof for *The Book*!

Contrast:

- Valid sentences are rare, and identifying them is undecidable!

- But almost-sure validity is the norm, and the decision problem is relatively easy ([Grandjean, 1983]: PSPACE-complete)

Beyond First-Order Logic

Observation:

- In standard mathematical logic, *first-order logic* is the Lingua Franca for foundational reasons.

- In finite-model theory, first-order logic is not priviliged. Many other logics are being studied, e.g., existental second-order logic, fixpoint logic, etc.

Question: Does the 0-1 Law extend beyond first-order logic?

0-1 Laws for Existential Second-Order Logic

Recall: ESO captures NP.

- No 0-1 law for ESO – can define *parity*

Kolaitis+V., 1987: Focus on first-order fragments of ESO

- $(\exists R_1)\ldots(\exists R_k)\varphi$, where φ is in a fragment of FO

Classification Project: classify fragments of FO that yields fragments of ESO with 0-1 laws.

- Kolaitis+V., 1987-8: positive results
- Pacholski+Szwast , 1989 and Le bars , 1998: negative results

0-1 Law for Infinitary Logic

Question: Why does FO have a *0-1 Law*?

Answer: [Kolaitis+V., 1990]: Because every sentence in FO have *finitely many* variables!

Corollary [K+V., 1990]: *Finite-variable infinitary logic* has a 0-1 Law!

Why Care? Because Finite-variable infinitary logic can express several fixpoint logics. $\Rightarrow$ 0-1 Law for Fixpoint Logics.

In Conclusion

Sad Truth: Most PhD dissertations are just not memorable.

In Contrast:

- Ron's dissertation is *memorable*!
- It is also *seminal* – the foundation stone for *Finite-Model Theory*
- It was an auspicious start to a highly distinguished research career.
- Most importantly, it has had a profound influence on my research career!